Incidence Matrices of t-Designs

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ABSTRACT

We point out a generalization of the matrix equation $NN^T = (r-\lambda)I + \lambda J$ to t-designs with $t>2$ and derive extensions of Fisher's, Connor's, and Mann's inequalities for block designs.

1. THE HIGHER INCIDENCE MATRICES

Given an incidence structure with v points X and b blocks $\mathcal{Q}$, the *incidence matrix* N is the $v \times b$ matrix whose rows are indexed by the elements x of X, whose columns are indexed by the elements A of $\mathcal{Q}$, and where the entry in row x and column A is 1 if x is incident with A (in which case we write $x \in A$) and 0 if x is not incident with A.

In 1949 [1], R. C. Bose observed that if N is the incidence matrix of a block design with parameters (v, b, r, k, λ), then

$$NN^T = (r-\lambda)I + \lambda J,$$

where I is the identity of order v and J is the $v \times v$ matrix of all ones. While it may seem strange to do so, the above equation can also be written

$$N_1 N_1^T = b_1^1 W_{11}^T W_{11} + b_2^0 W_{01}^T W_{01},$$

where $N_1 = N$, $b_1^1 = r - \lambda$ is the number of blocks which are incident with one given point but not incident with a second given point, $b_2^0 = \lambda$, $W_{11} = I$, and W_{01} is the $1 \times v$ matrix of all ones. It is the latter form which we extend.

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A *t-design* (or generalized Steiner system) $S_\lambda(t,k,v)$ is an incidence structure with v points, such that each block is incident with exactly k points, and in which any t points are simultaneously incident with exactly λ blocks. Here $t \leqslant k \leqslant v$, $\lambda \geqslant 1$. It is well known that a t-design is also an s-design for $s \leqslant t$; more precisely, the number of blocks which contain (or are incident with all of) a given s points is

$$b_s = \lambda \binom{v-s}{t-s} \bigg/ \binom{k-s}{t-s}.$$

A *block design* with parameters (v,b,r,k,λ) is an $S_\lambda(2,k,v)$ where $b = b_0$, $r = b_1$. Also well known (see, e.g., [6]) is that for $i + j \leqslant t$, the number of blocks of an $S_\lambda(t,k,v)$ which contain i given points but are not incident with any of a set of j other points is

$$b_i^j = \lambda \binom{v-i-j}{k-i} \bigg/ \binom{v-t}{k-t}. \tag{1}$$

With this notation, $b_i = b_i^0$. We write simply b for $b_0 = b_0^0$, i.e. the total number of blocks.

Let us introduce what might be called the higher incidence matrices of an incidence structure. For $i = 0,1,2,\ldots,$ let N_i denote the $\binom{v}{i} \times b$ matrix whose rows are indexed by the i-element subsets of points, whose columns are indexed by the blocks, and where the entry in row Y and column A is 1 if $Y \subseteq A$ and 0 otherwise. Thus N_0 is just a $1 \times b$ matrix of all ones, and if all blocks have the same size k, N_k is an identity matrix. For $0 \leqslant i \leqslant j \leqslant v$, we use $W_{ij} = W_{ij}(v)$ to denote the ith incidence matrix of the incidence structure whose blocks are all the j-element subsets of a v-set. Thus W_{ij} is a $\binom{v}{i} \times \binom{v}{j}$ matrix.

Proposition 1. *The incidence matrices $N_0, N_1, N_2, \ldots,$ of an $S_\lambda(t,k,v)$ satisfy*

$$N_e N_f^T = \sum_{i=0}^{\min\{e,f\}} b_{e+f-i}^i W_{ie}^T W_{if} \tag{2}$$

whenever $e + f \leqslant t$.

Proof. The matrix $N_e N_f^T$ has its rows indexed by e-element subsets E and columns indexed by f-element subsets F of the v points. Given E and F, the entry in row E, column F of $N_e N_f^T$ is the number of blocks which contain both E and F; and this number is $b_{e+f-\mu}$ in case $|E \cap F| = \mu$, say. The entry in row E, column F of $W_{ie}^T W_{if}$ is the number of i-subsets contained in both E and F, so the corresponding entry on the right-hand side of (2) is

$$\sum_{i=0}^{\min\{e,f\}} b_{e+f-i}^i \binom{\mu}{i}.$$

To see that this is equal to $b_{e+f-\mu}$, one can use Equation (1) and well-known combinatorial identities, but a perhaps better way is as follows: Choose a μ-subset M disjoint from $E \cup F$. The $b_{e+f-\mu}$ blocks which contain $E \cup F$ can be partitioned according to how they intersect M: the number which contain $E \cup F$ and exactly $\mu - i$ points of M is clearly $\binom{\mu}{i}$ times the number of blocks containing $E \cup F \cup S$ but disjoint from $M - S$ [here S is a $(\mu - i)$-subset of M], i.e. $\binom{\mu}{i} b_{e+f-i}^i$. The required equality is immediate. ■

2. FISHER'S INEQUALITY

Fisher's inequality asserts that $b \geq v$ for an $S_\lambda(2, k, v)$ with $v \geq k+1$. In 1969, A. Ya. Petrenjuk proved that

$$b \geq \binom{v}{2} \qquad \text{for any} \quad S_\lambda(4, k, v) \quad \text{with } v \geq k+2,$$

and conjectured the following theorem, which was proved by Ray-Chaudhuri and Wilson in [6]. By now, there are many proofs of this inequality, of which the following is one of the quickest.

THEOREM 1. *For an $S_\lambda(t, k, v)$ with $t \geq 2s$ and $v \geq k + s$, we have*

$$b \geq \binom{v}{s}.$$

Proof. Taking $e = f = s$ in Proposition 1, there results

$$N_s N_s^T = \sum_{i=0}^{s} b_{2s-i}^{i} W_{is}^T W_{is}.$$

The $\binom{v}{s} \times \binom{v}{s}$ matrices $b_{2s-i}^{i} W_{is}^T W_{is}$ are all positive semidefinite, and one of them, $b_s^s W_{ss}^T W_{ss} = b_s^s I$, is positive definite [$b_s^s > 0$ because $v \geqslant k + s$, e.g. from Equation (1)]. Hence $N_s N_s^T$, being the sum of a positive definite and positive semidefinite matrices, is positive definite and, in particular, is nonsingular. Over the reals, $\operatorname{rank}(N_s N_s^T) = \operatorname{rank}(N_s)$, so N_s has rank $\binom{v}{s}$ and this cannot, of course, exceed the number b of its columns. ■

3. MORE ON THE HIGHER INCIDENCE MATRICES

We summarize here some elementary relations between $N_0, N_1, N_2, \ldots$, and also introduce another family $\overline{N}_0, \overline{N}_1, \overline{N}_2, \ldots$, which will prove useful. Throughout this section we consider an incidence structure with v points and b blocks, and which is uniform, i.e., every block is incident with exactly the same number k of points. $\overline{N}_i$ will denote the $\binom{v}{i} \times b$ matrix whose rows are indexed by the i-subsets of points, whose columns are indexed by the blocks, and where the entry in row Y, column A is 1 if $A \cap Y = \varnothing$ (i.e., A is incident with no points of Y) and 0 otherwise. Again, $\overline{N}_0$ is the $1 \times b$ matrix of all ones. Similarly, we may consider $\overline{W}_{ij} = \overline{W}_{ij}(v)$.

PROPOSITION 2. *For* $0 \leqslant i \leqslant s \leqslant k$,

$$W_{is} N_s = \binom{k-i}{s-i} N_i, \tag{3}$$

$$W_{is} \overline{N}_s = \binom{v-k-i}{s-i} \overline{N}_i, \tag{4}$$

$$\overline{N}_s = \sum_{i=0}^{s} (-1)^i W_{is}^T N_i, \tag{5}$$

$$N_s = \sum_{i=0}^{s} (-1)^i W_{is}^T \overline{N}_i, \tag{6}$$

$$I = W_{ss} = \sum_{i=0}^{s} (-1)^i \overline{W}_{is}^T W_{is}. \tag{7}$$

Proof. To prove (3), consider the entry in a row labeled by an i-subset Y and a column labeled by a block A. On the left-hand side we get the number of s-subsets S with $Y \subseteq S \subseteq A$, and this number is $\binom{k-i}{s-i}$ if $Y \subseteq A$ and 0 otherwise, as claimed on the right-hand side of (3). The proof of (4) is similarly easy.

For the proof of (5), consider the entry in row S and column A on the right-hand side in the case that $|S \cap A| = \mu$; it is

$$\sum_{i=0}^{s} (-1)^i \binom{\mu}{i} = \begin{cases} 1 & \text{if} \quad \mu = 0, \\ 0 & \text{if} \quad 0 < \mu \leqslant s. \end{cases}$$

The proofs of (6) and (7) are similar. ∎

Let U_i denote the row space over the rationals $\mathbf{Q}$ of the matrix N_i, and $\overline{U}_i$ the row space of $\overline{N}_i$. Equations (3) and (4) show that $U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_k$ and $\overline{U}_0 \subseteq \overline{U}_1 \subseteq \overline{U}_2 \cdots \subseteq \overline{U}_k$. Equations (5) and (6) show that $\overline{U}_s \subseteq \bigcup_{i=0}^{s} U_i$ and $U_s \subseteq \bigcup_{i=0}^{s} \overline{U}_i$ for $s = 0, 1, \ldots, k$. We conclude that $U_s = \overline{U}_s$ for $s = 0, 1, \ldots, k$.

Note that our proof of Theorem 1 shows that

$$\dim(U_s) = \binom{v}{s}$$

if our incidence structure is an $S_\lambda(t, k, v)$ with $t \geqslant 2s$ and $v \geqslant k + s$.

4. CONNOR'S INEQUALITIES

In [2] (also see [3]) W. S. Connor developed a system of inequalities concerning the pairwise intersections $\mu_{ij} = |A_i \cap A_j|$ of m blocks $A_1, A_2, \ldots, A_m$ of an $S_\lambda(2, k, v)$. The $m \times m$ matrix

$$C = r(r-\lambda)I - r[\mu_{ij}] + \lambda kJ = r(r-\lambda)I - [r\mu_{ij} - \lambda k]$$

is called the characteristic matrix of the m blocks, and Connor's theorem is

$$\det(C) \geqslant 0, \tag{8i}$$

$$\det(C) = 0 \qquad \text{if} \quad m > b - v, \tag{8ii}$$

$$\text{If} \quad m = b - v, \qquad \frac{kr(r-\lambda)^{v-1}\det(C)}{r^m(r-\lambda)^m} \text{ is a perfect square.} \tag{8iii}$$

Parts (i) and (ii) of this theorem could also have been stated: The matrix $Q' = r(r-\lambda)I - rN^TN + \lambda kJ$ is positive semidefinite of rank $\leqslant b-v$, since the matrices C are exactly the principal submatrices of Q'. In fact, it turns out that $Q = [1/r(r-\lambda)]Q'$ is idempotent of rank exactly $b-v$; this is a consequence of the case $s=1$ of Theorem 2 below.

THEOREM 2. *Let P_s denote the matrix of the orthogonal projection from the vector space $\mathbb{Q}^b$ of b-tuples of rational numbers whose coordinates are indexed by the blocks of an $S_\lambda(t,k,v)$ with $t \geqslant 2s$ and $v \geqslant k+s$ onto U_s, the row space of the s th incidence matrix N_s. Then*

$$P_s = \sum_{i=0}^{s} (-1)^i \frac{\binom{k-i}{s-i}}{b_s^i} \overline{N}_i^T N_i. \tag{9}$$

Proof. We need to show that the matrix (call it P_s') on the right of (9) leaves invariant vectors in U_s and annihilates vectors in $U_s^\perp$. The latter is immediate from Proposition 2: $\mathbf{x} \in U_s^\perp$ implies $\mathbf{x} \in U_i^\perp = \overline{U}_i^\perp$ for $i \leqslant s$, so $\mathbf{x}\overline{N}_i^T = \mathbf{0}$ and then $\mathbf{x}P_s' = \mathbf{0}$. To establish the former assertion, we need only show $N_s P_s' = N_s$.

To this end, we first note the elementary

$$N_s \overline{N}_i^T = b_s^i \overline{W}_{is}^T,$$

and then compute, using Equations (3) and (7),

$$\begin{aligned} N_s P_s' &= \sum_{i=0}^{s} (-1)^i \frac{\binom{k-i}{s-i}}{b_s^i} N_s \overline{N}_i^T N_i = \sum_{i=0}^{s} (-1)^i \binom{k-i}{s-i} \overline{W}_{is}^T N_i \\ &= \sum_{i=0}^{s} (-1)^i \overline{W}_{is}^T W_{is} N_s = N_s. \end{aligned}$$ ■

Given λ, t, k, and v, consider the polynomials

$$p_s(x) = \sum_{i=0}^{s} (-1)^i \frac{\binom{k-i}{s-i}}{b_s^i} \binom{k-x}{i}, \qquad s = 0,1,2,\ldots,$$

where b_s^i is defined as in (1). The entry in row A and column B of P_s in (9) is $p_s(\mu)$, where $\mu = |A \cap B|$.

Under the hypothesis of Theorem 2, $Q_s = I - P_s$ is the matrix of the orthogonal projection onto $U_s^{\perp}$, and hence is idempotent of rank $b - \binom{v}{s}$. In particular, all principal submatrices of Q_s are positive semidefinite of rank $\leqslant b - \binom{v}{s}$, and we have

COROLLARY 1. *Let $A_1, A_2, \ldots, A_m$ be blocks of an $S_\lambda(t, k, v)$ with $t \geqslant 2s$ and $v \geqslant k + s$, and let $\mu_{ij} = |A_i \cap A_j|$. Then the $m \times m$ matrix $I - [p_s(\mu_{ij})]$ has nonnegative determinant, and is singular if*

$$m > b - \binom{v}{s}.$$

Since $p_1(\mu) = (r\mu - \lambda k)/r(r - \lambda)$, the case $s = 1$ of the above Corollary is just (8i) and (8ii).

Note that

$$p_s(k) = \frac{\binom{k}{s}}{b_s} = \frac{\binom{v}{s}}{b}. \tag{10}$$

The case $m = 1$ in Corollary 1 just asserts $1 - p_s(k) \geqslant 0$, and we recover Theorem 1. The case $m = 2$ shows that if two blocks A, B meet in μ points, then

$$|p_s(\mu)| \leqslant 1 - p_s(k). \tag{11}$$

In particular, (10) and (11) imply

COROLLARY 2. *If A, B are distinct blocks of an $S_\lambda(t, k, v)$ with $t \geqslant 2s$, $v \geqslant k + s$, and in which Theorem 1 is tight, i.e.*

$$b = \binom{v}{s},$$

then $|A \cap B|$ is a root of the polynomial $p_s(x)$.

The following corollary is the natural extension of H. B. Mann's generalization [5] of Fisher's inequality to block designs having "repeated blocks" (also see [4]).

COROLLARY 3. *If an $S_\lambda(t,k,v)$ with $t \geqslant 2s$ and $v \geqslant k+s$ has m distinct blocks which are incident with exactly the same set of k points, then*

$$b \geqslant m\binom{v}{s}.$$

Proof. With the notation of Corollary 1, $\mu_{ij} = k$ for $1 \leqslant i, j \leqslant m$ and the matrix $I - [p_s(\mu_{ij})]$ is

$$I - \binom{v}{s} b^{-1} J \qquad (\text{of order } m),$$

which has determinant

$$1 - \frac{m\binom{v}{s}}{b}. \qquad \blacksquare$$

5. A CONCLUDING REMARK

For completeness, we state here the analogue of (8iii). In general, if P is the matrix of the orthogonal projection of $\mathbb{Q}^b$ onto the row space U of a rank-n, $n \times b$ rational matrix N [P is just $N^T(NN^T)^{-1}N$] and $Q = I - P$, then $\det(NN^T)$ times the determinant of any $n \times n$ principal submatrix of P, or $(b-n)\times(b-n)$ principal submatrix of Q, is a rational square. Thus, in the case

$$m = b - \binom{v}{s}$$

in Corollary 1, we can add that $\det(N_s N_s^T)\det(I - [p_s(\mu_{ij})])$ is a square.

Proposition 3 below generalizes the well-known formula

$$\det(NN^T) = kr(r-\lambda)^{v-1} = kb_1^0(b_1^1)^{v-1},$$

which applies to the incidence matrix $N = N_1$ of a 2-design.

PROPOSITION 3. *If N_s is the sth incidence matrix of an $S_\lambda(t,k,v)$ with $t \geqslant 2s$ and $v \geqslant k+s$, then*

$$\det(N_s N_s^T) = \prod_{i=0}^{s} \left[\binom{k-i}{s-i} b_s^i\right]^{\binom{v}{i} - \binom{v}{i-1}}.$$

Proof. We use the fact that $N_s N_s^T$ and $N_s^T N_s$ have the same nonzero eigenvalues and multiplicities.

Let us write $M_i = N_i^T N_i$. By premultiplying Equation (2) by N_e^T, postmultiplying by N_f, and using (3), we have

$$M_e M_f = \sum_{i=0}^{\min\{e,f\}} \binom{k-i}{f-i}\binom{k-i}{e-i} b_{e+f-i}^i M_i \tag{12}$$

whenever $e+f \leqslant t$.

The row space U_i of N_i is also the row space of M_i. Let $V_0 = U_0$ and $V_i = U_i \cap U_{i-1}^{\perp}$ for $i = 1, 2, \dots, s$. If V denotes the b-dimensional space of all vectors with coordinates indexed by the blocks, then we have the decomposition $V = V_0 \oplus V_1 \oplus \cdots \oplus V_s \oplus U_s^{\perp}$ and V_i has dimension

$$\binom{v}{i} - \binom{v}{i-1}.$$

Clearly, vectors of $U_s^{\perp}$ are eigenvectors of value 0 for M_s. We claim vectors in V_e are eigenvectors of value $\binom{k-e}{s-e} b_s^e$ for M_s. The proposition will then be proved.

Given $\mathbf{x} \in V_e$, we have $\mathbf{x} \in U_e$, so we can write $\mathbf{x} = \mathbf{y} M_e$ for some $\mathbf{y}$. But $\mathbf{y} M_i = \mathbf{0}$ for $i < e$ because $\mathbf{x} \in U_i^{\perp}$ implies

$$\mathbf{0} = \mathbf{x} M_i = \mathbf{y} M_e M_i = \mathbf{y} M_i M_e$$

which means $\mathbf{y} M_i$ is in $U_e^{\perp}$ as well as in $U_i \subseteq U_e$. Premultiplying (12) by $\mathbf{y}$ and replacing f by s yields

$$\mathbf{x} M_s = \binom{k-e}{s-e} b_s^e$$

as required. ■

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